Math 2550 - Homework # 6

Coordinate systems in $\mathbb{R}^n$

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1. Show that the following vectors are linearly dependent. In addition, write one of the vectors as a linear combination of the other vectors.

   (a) In $\mathbb{R}^2$: $\vec{v} = \langle 2, 3\rangle$, $\vec{u} = \langle 1, \frac{3}{2}\rangle$

   (b) In $\mathbb{R}^2$: $\vec{v} = \langle 1, -1\rangle$, $\vec{u} = \langle 0, -3\rangle$, $\vec{w} = \langle 2, 1\rangle$

   (c) In $\mathbb{R}^3$: $\vec{v} = \langle 2, -1, 3\rangle$, $\vec{u} = \langle 4, 1, 2\rangle$, $\vec{w} = \langle 8, -1, 8\rangle$

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2. In $\mathbb{R}^2$ consider the vectors $\vec{a} = \langle 1, 1\rangle$, $\vec{b} = \langle -1, 1\rangle$

   (a) Show that $\vec{a}$, $\vec{b}$ are linearly independent. Conclude that they form a basis $\beta = [\vec{a}, \vec{b}]$ for $\mathbb{R}^2$.

   (b) Draw a picture of the two axes that $\vec{a}$ and $\vec{b}$ create. Label the $\vec{a}$-axis with $-3\vec{a}, -2\vec{a}, -\vec{a}, \vec{0}, \vec{a}, 2\vec{a}, 3\vec{a}$. Then do the same kind of labeling for the $\vec{b}$-axis. Then draw the grid that the axes create.

   (c) Draw a picture with $2\vec{a}$ and $\vec{b}$ and $2\vec{a} + \vec{b}$. Draw the parallelogram that is created.

   (d) Draw a picture with $-2\vec{a}$ and $-2\vec{b}$ and $-2\vec{a} - 2\vec{b}$. Draw the parallelogram that is created.

   (e) Find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle -1, 5\rangle$.

   (f) Find the coordinates $[\vec{w}]_\beta$ of $\vec{w} = \langle -3, -1\rangle$.

   (g) Show that β is an orthogonal basis, but not an orthonormal basis.

   (h) Use the Coordinate Dot Product Theorem to find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 10, \frac{1}{2}\rangle$.

   (i) Use the Coordinate Dot Product Theorem to find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 1, 2\rangle$.

   (j) Suppose you know that $[\vec{v}]_\beta = \langle 5, -4\rangle$. What is $\vec{v}$?

3. In $\mathbb{R}^2$ consider the vectors $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle 1, 0 \rangle$

(a) Show that $\vec{a}$, $\vec{b}$ are linearly independent. Conclude that they form a basis $\beta = [\vec{a}, \vec{b}]$ for $\mathbb{R}^2$.

(b) Draw a picture of the two axes that $\vec{a}$ and $\vec{b}$ create. Label the $\vec{a}$-axis with $-3\vec{a}, -2\vec{a}, -\vec{a}, \vec{0}, \vec{a}, 2\vec{a}, 3\vec{a}$. Then do the same kind of labeling for the $\vec{b}$-axis. Then draw the grid that the axes create.

(c) Draw a picture with $3\vec{a}$ and $\vec{b}$ and $3\vec{a} + \vec{b}$. Draw the parallelogram that is created.

(d) Draw a picture with $-\vec{a}$ and $2\vec{b}$ and $-\vec{a} + 2\vec{b}$. Draw the parallelogram that is created.

(e) Find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 1, 2 \rangle$.

(f) Find the coordinates $[\vec{w}]_\beta$ of $\vec{w} = \langle -1, 3 \rangle$.

(g) Show that β is not an orthogonal basis, and thus is also not an orthonormal basis.

(h) Suppose you know that $[\vec{v}]_\beta = \langle -3, 20 \rangle$. What is $\vec{v}$?

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4. In $\mathbb{R}^3$ consider the vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

(a) Show that $\vec{i}$, $\vec{j}$, $\vec{k}$ are linearly independent. Conclude that they form a basis $\beta = [\vec{i}, \vec{j}, \vec{k}]$ for $\mathbb{R}^3$.

(b) Draw a picture of $3\vec{i} + \vec{j}$. Show how it is created from the β vectors.

(c) Draw a picture of $-2\vec{i} + 2\vec{j} + \vec{k}$. Show how it is created from the β vectors.

(d) Find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle -1, 2, 1 \rangle$.

(e) Show that β is an orthonormal basis and an orthonormal basis.

(f) Use the Coordinate Dot Product Theorem to find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 6, 1, -5 \rangle$.

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5. In $\mathbb{R}^3$ consider the vectors $\vec{a} = \langle 1,1,0\rangle$, $\vec{b} = \langle -1,1,0\rangle$, $\vec{c} = \langle 0,0,1\rangle$

   (a) Show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent. Conclude that they form a basis $\beta = [\vec{a}, \vec{b}, \vec{c}]$ for $\mathbb{R}^3$

   (b) Suppose you know that $[\vec{v}]_\beta = \langle 3, 1, -4\rangle$. What is $\vec{v}$?

   (c) Find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 3,3,2\rangle$.

   (d) Show that β is an orthogonal basis, but not an orthonormal basis.

   (e) Use the Coordinate Dot Product Theorem to find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle 1,2,3\rangle$.

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6. In $\mathbb{R}^4$ consider the vectors $\vec{e}_1 = \langle 1,0,0,0\rangle$, $\vec{e}_2 = \langle 0,1,0,0\rangle$, $\vec{e}_3 = \langle 0,0,1,0\rangle$, $\vec{e}_4 = \langle 0,0,0,1\rangle$.

   (a) Show that $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_4$ are linearly independent. Conclude that they form a basis $\beta = [\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4]$ for $\mathbb{R}^4$. This basis is called the **standard basis** for $\mathbb{R}^4$.

   (b) Suppose you know that $[\vec{v}]_\beta = \langle -3, 1, -4, \pi\rangle$. What is $\vec{v}$?

   (c) Find the coordinates $[\vec{v}]_\beta$ of $\vec{v} = \langle \frac{2}{3}, 7, 5, -10\rangle$.

   (d) Show that β is an orthonormal basis.

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7. (a) Find an orthonormal basis for $\mathbb{R}^5$.
   [Hint: Look at problems 4 and 6. Do you see a pattern?]

   (b) Can you describe an orthonormal basis for $\mathbb{R}^n$?
